

MULTIPLE CRITERIA PROBLEMS OVER MINKOWSKI BALLS

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ABSTRACT. Under study are some vector optimization problems over the space of Minkowski balls, i. e., symmetric convex compact subsets in Euclidean space. A typical problem requires to achieve the best result in the presence of conflicting goals; e.g., given the surface area of a symmetric convex body \mathfrak{x} , we try to maximize the volume of \mathfrak{x} and minimize the width of \mathfrak{x} simultaneously.

INTRODUCTION

Vector optimization is another name for multiple criteria decision making. The mathematical technique of the field is rich but leaves much to be desired (for instance, see [1]– [3]). One of the reasons behind this is the fact that the classical areas of mathematics dealing with extremal problems pay practically no attention to the case of multiple criteria. So it seems reasonable to suggest attractive theoretical problems that involve many criteria. Some geometrical problems of the sort were considered in [4]. In this article we address similar problems over symmetric convex bodies, using the the same technique that stems from the classical Alexandrov’s approach to extremal problems of convex geometry [5].

1. CONVEX BODIES, BALLS, AND DUAL CONES

A *convex figure* is a compact convex set. A *convex body* is a solid convex figure. The *Minkowski duality* identifies a convex figure S in \mathbb{R}^N and its *support function* $S(z) := \sup\{(x, z) \mid x \in S\}$ for $z \in \mathbb{R}^N$. Considering the members of \mathbb{R}^N as singletons, we assume that \mathbb{R}^N lies in the set \mathcal{V}_N of all compact convex subsets of \mathbb{R}^N .

The Minkowski duality makes \mathcal{V}_N into a cone in the space $C(S_{N-1})$ of continuous functions on the Euclidean unit sphere S_{N-1} , the boundary of the unit ball \mathfrak{z}_N . The *linear span* $[\mathcal{V}_N]$ of \mathcal{V}_N is dense in $C(S_{N-1})$,

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bears a natural structure of a vector lattice and is usually referred to as the *space of convex sets*.

The study of this space stems from the pioneering breakthrough of Alexandrov in 1937 (see [5]) and the further insights of Radström, Hörmander, and Pinsker (see [6]).

A measure μ *linearly majorizes* or *dominates* a measure ν on S_{N-1} provided that to each decomposition of S_{N-1} into finitely many disjoint Borel sets U_1, \dots, U_m there are measures μ_1, \dots, μ_m with sum μ such that every difference $\mu_k - \nu|_{U_k}$ annihilates all restrictions to S_{N-1} of linear functionals over \mathbb{R}^N . In symbols, we write $\mu \gg_{\mathbb{R}^N} \nu$.

Reshetnyak proved in 1954 (see [7]) that

$$\int_{S_{N-1}} p d\mu \geq \int_{S_{N-1}} p d\nu$$

for each sublinear functional p on \mathbb{R}^N if $\mu \gg_{\mathbb{R}^N} \nu$. This gave an important trick for generating positive linear functionals over various classes of convex surfaces and functions. The converse of the Reshetnyak result was appeared in [8] and [9].

Alexandrov proved the unique existence of a translate of a convex body given its surface area function, thus completing the solution of the Minkowski problem. Each surface area function is an *Alexandrov measure*. So we call a positive measure on the unit sphere which is supported by no great hypersphere and which annihilates singletons.

Each Alexandrov measure is a translation-invariant additive functional over the cone \mathcal{V}_N . The cone of positive translation-invariant measures in the dual $C'(S_{N-1})$ of $C(S_{N-1})$ is denoted by \mathcal{A}_N .

Given $\mathfrak{x}, \mathfrak{y} \in \mathcal{V}_N$, the record $\mathfrak{x} =_{\mathbb{R}^N} \mathfrak{y}$ means that \mathfrak{x} and \mathfrak{y} are equal up to translation or, in other words, are translates of one another. So, $=_{\mathbb{R}^N}$ is the associate equivalence of the preorder $\geq_{\mathbb{R}^N}$ on \mathcal{V}_N of the possibility of inserting one figure into the other by translation.

The sum of the surface area measures of \mathfrak{x} and \mathfrak{y} generates the unique class $\mathfrak{x} \# \mathfrak{y}$ of translates which is referred to as the *Blaschke sum* of \mathfrak{x} and \mathfrak{y} . There is no need in discriminating between a convex figure, the coset of its translates in $\mathcal{V}_N / \mathbb{R}^N$, and the corresponding measure in \mathcal{A}_N .

Let $C(S_{N-1}) / \mathbb{R}^N$ stand for the factor space of $C(S_{N-1})$ by the subspace of all restrictions of linear functionals on \mathbb{R}^N to S_{N-1} . Let $[\mathcal{A}_N]$ be the space $\mathcal{A}_N - \mathcal{A}_N$ of translation-invariant measures, in fact, the linear span of the set of Alexandrov measures.

$C(S_{N-1})/\mathbb{R}^N$ and $[\mathcal{A}_N]$ are made dual by the canonical bilinear form

$$\langle f, \mu \rangle = \frac{1}{N} \int_{S_{N-1}} f d\mu$$

$$(f \in C(S_{N-1})/\mathbb{R}^N, \mu \in [\mathcal{A}_N]).$$

For $\mathfrak{x} \in \mathcal{V}_N/\mathbb{R}^N$ and $\mathfrak{y} \in \mathcal{A}_N$, the quantity $\langle \mathfrak{x}, \mathfrak{y} \rangle$ coincides with the *mixed volume* $V_1(\mathfrak{y}, \mathfrak{x})$.

Consider the set $\text{Sym } \mathcal{V}_N$ of centrally symmetric cosets of convex compact sets. Clearly, a translation-invariant linear functional f is positive over $\text{Sym } \mathcal{V}_N$ if and only if the *symmetrization* $\text{Sym}(f)$ is positive over \mathcal{V}_N . Here $\text{Sym}(f)$ is the dual of the descent of the even part operator on the factor-space, since the symmetrization of a measure is the dual of the even part operator over $C(S_{N-1})$. We will denote the even part operator, its descent and dual by the same symbol $\text{Sym}(\cdot)$.

Given a cone K in a vector space X in duality with another vector space Y , the *dual* of K is

$$K^* := \{y \in Y \mid (\forall x \in K) \langle x, y \rangle \geq 0\}.$$

To a convex subset U of X and $\bar{x} \in U$ there corresponds

$$U_{\bar{x}} := \text{Fd}(U, \bar{x}) := \{h \in X \mid (\exists \alpha \geq 0) \bar{x} + \alpha h \in U\},$$

the *cone of feasible directions* of U at \bar{x} .

Let $\bar{\mathfrak{x}} \in \mathcal{A}_N$. Then the dual $\mathcal{A}_{N, \bar{\mathfrak{x}}}^*$ of the cone of feasible directions of \mathcal{A}_N at $\bar{\mathfrak{x}}$ may be represented as follows

$$\mathcal{A}_{N, \bar{\mathfrak{x}}}^* = \{f \in \mathcal{A}_N^* \mid \langle \bar{\mathfrak{x}}, f \rangle = 0\}.$$

The description of the dual of the feasible cones are well known (see [10, Proposition 4.3].

Let \mathfrak{x} and \mathfrak{y} be convex figures. Then

- (1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \mathcal{V}_N^* \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;
- (2) *If $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ then $\mu(\mathfrak{x}) \gg_{\mathbb{R}^N} \mu(\mathfrak{y})$;*
- (3) $\mathfrak{x} \geq_{\mathbb{R}^2} \mathfrak{y} \leftrightarrow \mu(\mathfrak{x}) \gg_{\mathbb{R}^2} \mu(\mathfrak{y})$;
- (4) *If $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in \mathcal{V}_{N, \bar{\mathfrak{x}}}^*$ then $\mathfrak{y} =_{\mathbb{R}^N} \bar{\mathfrak{x}}$ for $\bar{\mathfrak{x}} \in \mathcal{V}_N$.*

From this the dual cones are available in the case of Minkowski balls.

Let \mathfrak{x} and \mathfrak{y} be convex figures. Then

- (1) $\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \in \text{Sym } \mathcal{V}_N^* \leftrightarrow \text{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^N} \text{Sym}(\mu(\mathfrak{y}))$;
- (2) *If $\mathfrak{x} \geq_{\mathbb{R}^N} \mathfrak{y}$ then $\text{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^N} \text{Sym}(\mu(\mathfrak{y}))$;*
- (3) $\text{Sym}(\mathfrak{x}) \geq_{\mathbb{R}^2} \text{Sym}(\mathfrak{y}) \leftrightarrow \text{Sym}(\mu(\mathfrak{x})) \gg_{\mathbb{R}^2} \text{Sym}(\mu(\mathfrak{y}))$;
- (4) *If $\mu(\mathfrak{y}) - \mu(\bar{\mathfrak{x}}) \in (\text{Sym } \mathcal{V})_{N, \bar{\mathfrak{x}}}^*$ then $\text{Sym}(\mathfrak{y}) =_{\mathbb{R}^N} \bar{\mathfrak{x}}$ for $\bar{\mathfrak{x}} \in (\text{Sym } \mathcal{V})_N$.*

2. ALEXANDROV'S APPROACH TO THE URYSOHN PROBLEM

Alexandrov observed that the gradient of $V(\cdot)$ at \mathfrak{x} is proportional to $\mu(\mathfrak{x})$ and so minimizing $\langle \cdot, \mu \rangle$ over $\{V = 1\}$ will yield the equality $\mu = \mu(\mathfrak{x})$ by the Lagrange multiplier rule. But this idea fails since the interior of \mathcal{V}_N is empty. The fact that DC-functions are dense in $C(S_{N-1})$ is not helpful at all.

Alexandrov extended the volume to the positive cone of $C(S_{N-1})$ by the formula $V(f) := \langle f, \mu(\text{co}(f)) \rangle$ with $\text{co}(f)$ the envelope of support functions below f . He also observed that $V(f) = V(\text{co}(f))$. The ingenious trick settled all for the Minkowski problem. This was done in 1938 but still is one of the summits of convexity.

In fact, Alexandrov suggested a functional analytical approach to extremal problems for convex surfaces. To follow it directly in the general setting is impossible without the above description of the dual cones. The obvious limitations of the Lagrange multiplier rule are immaterial in the case of convex programs. It should be emphasized that the classical isoperimetric problem is not a Minkowski convex program in dimensions greater than 2. The convex counterpart is the Urysohn problem of maximizing volume given integral breadth [11]. The constraints of inclusion type are convex in the Minkowski structure, which opens way to complete solution of new classes of Urysohn-type problems.

The External Urysohn Problem: Among the convex figures, circumscribing \mathfrak{x}_0 and having integral breadth fixed, find a convex body of greatest volume.

A feasible convex body $\bar{\mathfrak{x}}$ is a solution to the external Urysohn problem if and only if there are a positive measure μ and a positive real $\bar{\alpha} \in \mathbb{R}_+$ satisfying

- (1) $\bar{\alpha}\mu(\mathfrak{z}_N) \gg_{\mathbb{R}^N} \mu(\bar{\mathfrak{x}}) + \mu;$
- (2) $V(\bar{\mathfrak{x}}) + \frac{1}{N} \int_{S_{N-1}} \bar{\mathfrak{x}} d\mu = \bar{\alpha}V_1(\mathfrak{z}_N, \bar{\mathfrak{x}});$
- (3) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ for all z in the support of μ , i. e. $z \in \text{spt}(\mu)$.

If $\mathfrak{x}_0 = \mathfrak{z}_{N-1}$ then $\bar{\mathfrak{x}}$ is a *spherical lens* and μ is the restriction of the surface area function of the ball of radius $\bar{\alpha}^{1/(N-1)}$ to the complement of the support of the lens to S_{N-1} .

If \mathfrak{x}_0 is an equilateral triangle then the solution $\bar{\mathfrak{x}}$ looks as in Fig. 1:

$\bar{\mathfrak{x}}$ is the union of \mathfrak{x}_0 and three congruent slices of a circle of radius $\bar{\alpha}$ and centers O_1-O_3 , while μ is the restriction of $\mu(\mathfrak{z}_2)$ to the subset of S_1 comprising the endpoints of the unit vectors of the shaded zone.

Fig. 2 presents the general solution of the internal Urysohn problem inside a triangle in the class of Minkowski balls.

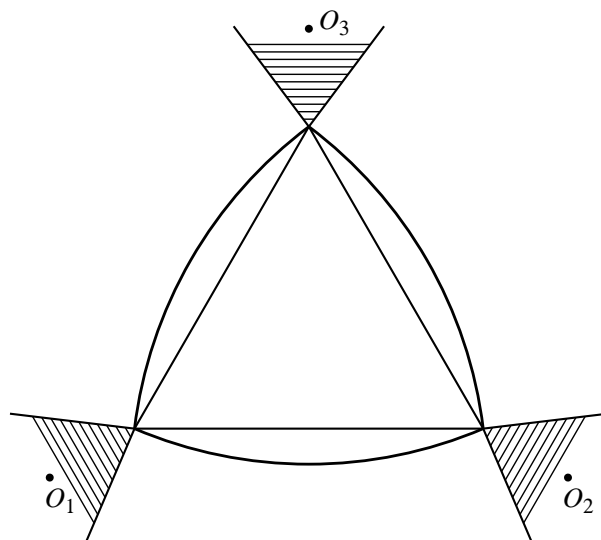


Fig. 1

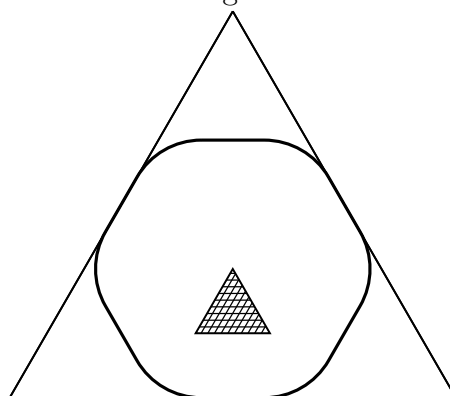


Fig. 2

3. PARETO'S APPROACH TO VECTOR OPTIMIZATION OVER MINKOWSKI BALLS

Consider a bunch of economic agents each of which intends to maximize his own income. The *Pareto efficiency principle* asserts that as an effective agreement of the conflicting goals it is reasonable to take any state in which nobody can increase his income in any way other than diminishing the income of at least one of the other fellow members. Formally speaking, this implies the search of the maximal elements of the set comprising the tuples of incomes of the agents at every state; i.e., some vectors of a finite-dimensional arithmetic space endowed with the coordinatewise order. Clearly, the concept of Pareto optimality was already abstracted to arbitrary ordered vector spaces.

By way of example, consider a few multiple criteria problems of isoperimetric type. For more detail, see [4].

Vector Isoperimetric Problem Over Minkowski Balls: Given are some convex bodies η_1, \dots, η_M . Find a symmetric convex body \mathfrak{x} encompassing a given volume and minimizing each of the mixed volumes $V_1(\mathfrak{x}, \eta_1), \dots, V_1(\mathfrak{x}, \eta_M)$. In symbols,

$$\mathfrak{x} \in \text{Sym}(\mathcal{A}_N); \quad \widehat{p}(\mathfrak{x}) \geq \widehat{p}(\bar{\mathfrak{x}}); \quad (\langle \eta_1, \mathfrak{x} \rangle, \dots, \langle \eta_M, \mathfrak{x} \rangle) \rightarrow \inf.$$

Clearly, this is a Slater regular convex program in the Blaschke structure.

Each Pareto-optimal solution $\bar{\mathfrak{x}}$ of the vector isoperimetric problem has the form

$$\bar{\mathfrak{x}} = \alpha_1 \text{Sym}(\eta_1) + \dots + \alpha_m \text{Sym}(\eta_m),$$

where $\alpha_1, \dots, \alpha_m$ are positive reals.

Internal Urysohn Problem with Flattening Over Minkowski Balls: Given are some convex body $\mathfrak{x}_0 \in \text{Sym } \mathcal{V}_N$ and some flattening direction $\bar{z} \in S_{N-1}$. Considering $\mathfrak{x} \subset \mathfrak{x}_0$ of fixed integral breadth, maximize the volume of \mathfrak{x} and minimize the breadth of \mathfrak{x} in the flattening direction: $\mathfrak{x} \in \text{Sym } \mathcal{V}_N; \mathfrak{x} \subset \mathfrak{x}_0; \langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle; (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \longrightarrow \inf$.

For a feasible symmetric convex body $\bar{\mathfrak{x}}$ to be Pareto-optimal in the internal Urysohn problem with the flattening direction \bar{z} over Minkowski balls it is necessary and sufficient that there be positive reals α and β together with a convex figure \mathfrak{x} satisfying

$$\begin{aligned} \mu(\bar{\mathfrak{x}}) &= \text{Sym}(\mu(\mathfrak{x})) + \alpha \mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}}); \\ \bar{\mathfrak{x}}(z) &= \mathfrak{x}_0(z) \quad (z \in \text{spt}(\mu(\mathfrak{x})). \end{aligned}$$

By way of illustration we will derive the optimality criterion in somewhat superfluous detail.

Note firstly that the internal Urysohn problem with flattening over Minkowski balls may be rephrased in $C(S_{N-1})$ as the following two-objective program

$$\begin{aligned} \mathfrak{x} &\in \text{Sym } \mathcal{V}_N; \\ \max\{\mathfrak{x}(z) - \mathfrak{x}_0(z) \mid z \in S_{N-1}\} &\leq 0; \\ \langle \mathfrak{x}, \mathfrak{z}_N \rangle &\geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle; \\ (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) &\longrightarrow \inf. \end{aligned}$$

The problem of Pareto optimization reduces to the scalar program

$$\begin{aligned} \mathfrak{x} &\in \text{Sym } \mathcal{V}_N; \\ \max\{\max\{\mathfrak{x}(z) - \mathfrak{x}_0(z) \mid z \in S_{N-1}\}, \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle - \langle \mathfrak{x}, \mathfrak{z}_N \rangle\} &\leq 0; \\ \max\{-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})\} &\longrightarrow \inf. \end{aligned}$$

The last program is Slater-regular and so we may apply the *Lagrange principle*. In other words, the value of the program under consideration coincides with the value of the free minimization problem for an appropriate Lagrangian:

$$\begin{aligned} & \mathfrak{x} \in \text{Sym } \mathcal{V}_N; \\ & \max\{-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})\} + \gamma \max\{\max\{\mathfrak{x}(z) - \mathfrak{x}_0(z) \mid z \in S_{N-1}\}, \\ & \quad \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle - \langle \mathfrak{x}, \mathfrak{z}_N \rangle\} \longrightarrow \inf. \end{aligned}$$

Here γ is a positive Lagrange multiplier.

We are left with differentiating the Lagrangian along the feasible directions and appealing to the description of the dual cones. Note in particular that the relation

$$\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \quad (z \in \text{spt}(\mu(\mathfrak{x})))$$

is the *complementary slackness condition* standard in mathematical programming. The proof of the optimality criterion for the Urysohn problem with flattening over Minkowski balls complete.

Rotational Symmetry: Assume that a plane convex figure $\mathfrak{x}_0 \in \mathcal{V}_2$ has the symmetry axis $A_{\bar{z}}$ with generator \bar{z} . Assume further that \mathfrak{x}_{00} is the result of rotating \mathfrak{x}_0 around the symmetry axis $A_{\bar{z}}$ in \mathbb{R}^3 . Consider the problem:

$$\begin{aligned} & \mathfrak{x} \in \mathcal{V}_3; \\ & \mathfrak{x} \text{ is a convex body of rotation around } A_{\bar{z}}; \\ & \mathfrak{x} \supset \mathfrak{x}_{00}; \quad \langle \mathfrak{z}_N, \mathfrak{x} \rangle \geq \langle \mathfrak{z}_N, \bar{\mathfrak{x}} \rangle; \\ & (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \longrightarrow \inf. \end{aligned}$$

Each Pareto-optimal solution is the result of rotating around the symmetry axis a Pareto-optimal solution of the plane internal Urysohn problem with flattening in the direction of the axis.

The External Urysohn Problem with Flattening Over Minkowski Balls: Given are some convex body $\mathfrak{x}_0 \in \mathcal{V}_N$ and flattening direction $\bar{z} \in S_{N-1}$. Considering Minkowski balls $\mathfrak{x} \supset \mathfrak{x}_0$ of fixed integral breadth, maximize volume and minimize breadth in the flattening direction: $\mathfrak{x} \in \text{Sym } \mathcal{V}_N; \mathfrak{x} \supset \mathfrak{x}_0; \langle \mathfrak{x}, \mathfrak{z}_N \rangle \geq \langle \bar{\mathfrak{x}}, \mathfrak{z}_N \rangle; (-p(\mathfrak{x}), b_{\bar{z}}(\mathfrak{x})) \longrightarrow \inf.$

For a feasible convex body $\bar{\mathfrak{x}}$ to be a Pareto-optimal solution of the external Urysohn problem with flattening over Minkowski balls it is necessary and sufficient that there be positive reals α and β together

with a convex figure \mathfrak{x} satisfying

$$\begin{aligned}\mu(\bar{\mathfrak{x}}) + \text{Sym}(\mu(\mathfrak{x})) &\gg_{\mathbb{R}^N} \alpha\mu(\mathfrak{z}_N) + \beta(\varepsilon_{\bar{z}} + \varepsilon_{-\bar{z}}); \\ V(\bar{\mathfrak{x}}) + V_1(\text{Sym}(\mathfrak{x}), \bar{\mathfrak{x}}) &= \alpha V_1(\mathfrak{z}_N, \bar{\mathfrak{x}}) + 2N\beta b_{\bar{z}}(\bar{\mathfrak{x}}); \\ \bar{\mathfrak{x}}(z) &= \mathfrak{x}_0(z) \quad (z \in \text{spt}(\mu(\mathfrak{x}))).\end{aligned}$$

REFERENCES

- [1] Kusraev A. G. and Kutateladze S. S., *Subdifferentials: Theory and Applications*. Dordrecht: Kluwer Academic Publishers, 1995.
- [2] Figueira J., Greco S., and Ehrgott M. *Multiple Criteria Decision Analysis. State of the Art Surveys*. Boston: Springer Science + Business Media, Inc. (2005).
- [3] Boţ R. I., Grad S.-M., and Wanka G. *Duality in Vector Optimization*. Berlin–Heidelberg: Springer-Verlag (2009).
- [4] Kutateladze S. S. “Multiobjective Problems of Convex Geometry,” *Siberian Math. J.*, 50:5, 887–897 (2009).
- [5] Alexandrov A. D. *Selected Works. Part 1: Selected Scientific Papers*. Ed. by Reshetnyak Yu. G. and Kutateladze S. S. Amsterdam: Gordon and Breach, 1996.
- [6] Kutateladze S. S. and Rubinov A. M. “The Minkowski Duality and Its Applications,” *Russian Math. Surveys*, 1972, 27:3, 137–191 (1972).
- [7] Reshetnyak Yu. G. *On the Length and Swerve of a Curve and the Area of a Surface*, (Ph. D. Thesis, unpublished), Leningrad State University (1954).
- [8] Kutateladze S. S. “Positive Linear Functionals over Convex Surfaces,” *Optimal Planning*, 14, 80–87 (1969).
- [9] Kutateladze S. S. “Positive Minkowski-Linear Functionals over Convex Surfaces,” *Soviet Math. Dokl.*, 11:3, 767–769 (1970).
- [10] Kutateladze S. S. “Interaction of Order and Convexity,” *J. Appl. Indust. Math.*, 1:4, 399–405 (2007).
- [11] Urysohn P. S. “Interdependence Between the Integral Breadth and Volume of a Convex Body,” *Mat. Sb.*, 31:3, 477–486 (1924).

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